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Remarks on the Independence of the Free Energy from Crystalline Boundary Conditions in the Two-Dimensional One-Component Plasma

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We study the two-dimensional one-component plasma. We show that given a bound on the one-particle correlation functions in the thermodynamic limit the canonical free energy is independent or free of the Dobrushin-type boundary conditions obtained by putting outside the vessel a regular configuration of fixed charges.

KEY WORDS: Statistical mechanics; two-dimensional classical Coulomb systems; one-component plasma; Dobrushin boundary conditions; free energy; two-dimensional crystals; jellium.

1. INTRODUCTION

In this paper we study a classical mechanical system of N discrete point charges immersed in a continuous uniform (i.e., homogeneous) neutralizing background in a two-dimensional domain. This is a model which has been studied extensively in plasma physics under the name "two-dimensional one-component plasma" (or short "OCP-plasma")—see, e.g., Ref. 1 and references therein. It is also known under the name of two-dimensional "jellium model,"[(2,3)] and has been applied, e.g., in metal physics, astrophysics, and to the study of electrolyte solutions. A problem which is often discussed in this model (and the corresponding two-dimensional versions) is the existence of a crystalline phase—see, e.g., Refs. 4–6 (in one dimension

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this has been proven[7]). In such discussions the possibility of a dependence of the canonical free energy in the thermodynamic limit of the different boundary conditions has been advanced—see, e.g., Ref 2. In the case of lattice models with short-range interactions, in all dimensions, the independence of the canonical free energy density on boundary conditions is well known—see, e.g., Ref. 8. This independence result is also known for certain continuous short-range classical statistical models of gases,[9] for the models studied in connection with two-dimensional Euclidean quantum fields—see, e.g., Refs 10–12—and in short-range models of quantum statistical mechanics—see, e.g., Ref. 13. In all these cases the short-range nature of forces plays an essential role. The fact that the two-dimensional jellium model has long-range (logarithmically increasing!) forces causes problems in this respect. The existence of the thermodynamic limit for the canonical free energy density with free (i.e., open) boundary conditions has been proven by Sari and Merlini[3] (the three-dimensional case had been treated by Lieb and Narnhofer[2]). However, the dependence or independence of boundary conditions has not been proven, to our knowledge.

In the present paper we make a step in this direction by showing that independence within a class of boundary conditions follows from a bound on the one-particle correlation function. The boundary conditions we consider are of the type of a fixed crystalline configuration of point charges outside a bounded region, with neutralizing homogeneous background. Such boundary conditions being an analog of the boundary conditions with all spin + 1 (or − 1) outside a finite region in the case of the Ising model, one might expect a symmetry breakdown of the equilibrium Gibbs state, similarly as in the Ising model. This is still an open question for the jellium model.

Our independence result implies that in order to find possible phase transitions one should investigate the analytic properties of the (unique) free energy in the thermodynamic limit. (See also the discussion in Refs. 14–18.)

We shall now describe briefly the models we consider and our results. We consider a bounded domain Λ and the classical interaction energy given by

$$H_\Lambda(\mathbf{x}) = \sum_{\substack{i<j \\ 1}}^{N} \varphi(x_i, x_j) - \sum_{i=1}^{N} \rho \int_\Lambda \varphi(x, x_i)\, dx + \frac{1}{2} \rho^2 \int_\Lambda \int_\Lambda \varphi(x, y)\, dx\, dy \quad (1.1)$$

where $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_N)$, x_i, $i = 1, \ldots, N$ being the coordinates of N positive unit charges. $\rho \equiv N/|\Lambda|$ is the constant background density (so that $\rho|\Lambda| = N$ is the negative charge in Λ, with volume $|\Lambda|$, balancing exactly the positive charge N in Λ). φ is the two-dimensional Coulomb potential, i.e., the kernel of Δ^{-1}, Δ being the Laplacian (i.e., in the case of free boundary conditions: $\varphi(x, y) \equiv -(1/2\pi)\ln|x-y|$, with $|\ |$ the distance of $\mathbb{R}^2$). The first term in H_Λ represents thus the interaction between the positive charges, the second the one between the positive charges and the background, and the third term the one given by the self-interaction of the background. It has been shown in Ref. 3 that H_Λ is lower bounded. The corresponding canonical free energy (density)

$$f_\Lambda \equiv -\frac{1}{\beta|\Lambda|}\ln Q_\Lambda, \qquad Q_\Lambda \equiv \frac{1}{N!}\int_{\Lambda^N}\exp[-\beta H_\Lambda(\mathbf{x})]\,d\mathbf{x}$$

is well defined. Let now Λ_0 be a larger bounded domain such that $\rho|\Lambda_0 - \Lambda|$ is again an integer. We place charges $+1$ at each point of some discrete subset Y of $\Lambda_0 - \Lambda$ (in the most interesting applications Y will be a lattice), so that the total positive charge in $\Lambda_0 - \Lambda$ is precisely $\rho|\Lambda_0 - \Lambda|$. We look at these charges at *fixed* position in $\Lambda_0 - \Lambda$ as a "Dobrushin boundary condition" for the configuration inside Λ. We then define

$$H_{\Lambda,\Lambda_0} \equiv H_\Lambda + V_{\Lambda,\Lambda_0}$$

with

$$V_{\Lambda,\Lambda_0} \equiv \sum_{i=1}^{N}\sum_{y\in Y}\varphi(y,x_i) - \rho\sum_{y\in Y}\int_\Lambda \varphi(x,y)\,dx - \rho\sum_{i=1}^{N}\int_{\Lambda_0-\Lambda}\varphi(x,x_i)\,dx + \rho^2\int_\Lambda\left[\int_{\Lambda_0-\Lambda}\varphi(x,y)\,dy\right]dx$$

V_{Λ,Λ_0} gives the interaction between the charges in Λ and in Y, and precisely the first term the one of the positive charges, the second for the positive charges in Y and the background in Λ, the third the one of the positive charges in Λ and the background in $\Lambda_0 - \Lambda$, and the fourth between the backgrounds in Λ and $\Lambda_0 - \Lambda$. Note that we do not consider in V_{Λ,Λ_0} the self-energy of the system in $\Lambda_0 - \Lambda$. Define the canonical free energy given by H_{Λ,Λ_0} as follows: $f_{\Lambda,\Lambda_0} \equiv -(\beta|\Lambda|)^{-1}\ln Q_{\Lambda,\Lambda_0}$, with Q_{Λ,Λ_0} the canonical partition function associated with H_{Λ,Λ_0}, i.e.,

$$Q_{\Lambda,\Lambda_0} \equiv (N!)^{-1}\int_{\Lambda^N}\exp[-\beta H_{\Lambda,\Lambda_0}(\mathbf{x})]\,d\mathbf{x}$$

We call the system described by (H_Λ, f_Λ) "the system with free boundary condition" and the one described by $(H_{\Lambda,\Lambda_0}, f_{\Lambda,\Lambda_0})$ "the system with Dobrushin boundary condition." In the present paper we study the quantities f_Λ and f_{Λ,Λ_0} and prove that, roughly speaking, for suitable choices of Y, the thermodynamic limits of these quantities coincide, i.e., the canonical free energy is the same for free and Dobrushin boundary conditions provided one has a certain bound on correlation functions.[18] We note that this implies the uniqueness (in the sense of independence from the mentioned boundary conditions) of the thermal pressure. Whether the kinetic pressure is also unique is not discussed here. We also do not discuss the problem of whether a Dobrushin boundary condition may induce a symmetry breaking of the state as $|\Lambda| \to \infty$ at low temperature (large β) giving rise to a "crystal." However, our result can be combined with an argument given recently[15] for the two-dimensional $1/r$ potential to obtain some results concerning the domain of nonexistence of long-range positional crystalline order in the two-dimensional jellium model.[19] We would like to mention that the possibility of the breaking of rotation symmetry (directional ordering) of the state has also been discussed recently.[16,17]

Let us now briefly give a few details on the content of the sections:

In Section 2 we study the jellium model in a square Λ, Λ_0 being a concentric square and the Dobrushin boundary condition Y in $\Lambda_0 - \Lambda$ being a square lattice with generators parallel to the sides of Λ.

In Section 3 we study the jellium model in a circle Λ, Λ_0 being a concentric circle and the Dobrushin boundary condition Y being a regular configuration in $\Lambda_0 - \Lambda$ given by equidistant points on regularly spaced circumferences concentric with Λ, Λ_0.

In Section 4 we study the jellium model in a square Λ with periodic boundary conditions in one direction, i.e., a jellium model on a cyclinder. Λ_0 is a larger cylinder with the same basis and Y is a square lattice with generators parallel to the sides of Λ.

In all cases the method of proof consists in showing that the average of $|\Lambda|^{-1} V_{\Lambda,\Lambda_0}$ with respect to the canonical Gibbs measures given by H_Λ, respectively, H_{Λ,Λ_0}, converge to zero as $\Lambda, \Lambda_0 \uparrow \mathbb{R}^2$. This is shown by explicit computations using the Coulomb potential and the symmetries of the models, together with estimates on one-particle canonical correlation functions discussed.[18]

2. THE COULOMB PLASMA IN RECTANGULAR DOMAINS

Let Λ be a bounded domain in $\mathbb{R}^2$ and let N be a fixed integer. Let H be the following function of $\mathbf{x} \equiv (x_1, \ldots, x_N)$, where $x_i \in \Lambda$, $i = 1, \ldots, N$:

$$H_\Lambda(\mathbf{x}) \equiv \sum_{\substack{i<j \\ 1}}^{N} \varphi(x_i, x_j) - \rho \sum_{i=1}^{N} \int_\Lambda \varphi(x, x_i)\, dx + \frac{1}{2} \rho^2 \int_\Lambda \int_\Lambda \varphi(x, y)\, dx\, dy \quad (2.1)$$

where $\rho \equiv N/|\Lambda|$, $|\Lambda|$ being the volume of Λ, and $\varphi(x, y) \equiv -(1/2\pi) \ln|x - y|$ is the Coulomb potential between x, y.

H_Λ is lower bounded, in fact it was proven in Ref. 3 that

$$H_\Lambda \geqslant -N\left[\tfrac{3}{8} + \tfrac{1}{4}\ln(\pi\rho)\right] \quad (2.2)$$

H_Λ is the interaction for the system of N charged particles interacting in Λ between themselves and a fixed background described in Section 1. By (2.2) for any $\beta \geqslant 0$ the function of Λ (and β, N)

$$Q_\Lambda \equiv \frac{1}{N!} \int_{\Lambda^N} \exp[-\beta H_\Lambda(\mathbf{x})]\, d\mathbf{x} \tag{2.3}$$

with $d\mathbf{x} \equiv \prod_{i=1}^N dx_i$ is well defined. Q_Λ is the canonical partition function associated with the system. As shown in Ref. 3, we have $Q_\Lambda > 0$; in fact

$$Q_\Lambda \geqslant \exp[-\rho \ln \rho + \beta\rho\alpha(\rho)] \tag{2.4}$$

with $\alpha(\rho)$ a bounded measurable function of ρ. Thus the canonical free energy

$$f_\Lambda \equiv -(\beta|\Lambda|)^{-1} \ln Q_\Lambda \tag{2.5}$$

is well defined.

Let Λ_0 be a bounded measurable domain such that $\Lambda_0 \supset \Lambda$ and such that $|\Lambda_0 - \Lambda|\rho$ is an integer. Let $\mathbf{e} \equiv (e_1, e_2)$ with e_i for $i = 1, 2$ fixed linearly independent vectors in $\mathbb{R}^2$, and let $\mathbb{Z}_\mathbf{e}$ be the lattice with integer coordinates spanned by the e_i, i.e.,

$$\mathbb{Z}_\mathbf{e} \equiv \left\{ \sum_{i=1}^{2} l_i e_i,\ l_i \in \mathbb{Z} \right\}$$

Denote $(\Lambda_0 - \Lambda)_d$ the points of $\mathbb{Z}_\mathbf{e}$ lying in $\Lambda - \Lambda_0$, i.e., $(\Lambda_0 - \Lambda)_d \equiv (\Lambda_0 - \Lambda) \cap \mathbb{Z}_\mathbf{e}$, and define

$$\begin{aligned} V_{\Lambda,\Lambda_0} \equiv {} & \sum_{i=1}^{N} \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \varphi(x_l, x_i) - \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \rho \int_\Lambda \varphi(x, x_l)\, dx \\ & - \sum_{i=1}^{N} \rho \int_{\Lambda_0 - \Lambda} \varphi(x, x_i)\, dx + \rho^2 \int_\Lambda \left[\int_{\Lambda_0 - \Lambda} \varphi(x, y)\, dy \right] dx \end{aligned} \tag{2.6}$$

As described in Section 1, V_{Λ,Λ_0} gives the total interaction of the charges in Λ and $(\Lambda_0 - \Lambda)_d$, except for the total self-interaction in $(\Lambda_0 - \Lambda)_d$. Similarly as in the proof mentioned above of the lower bound on H_Λ one proves that $\mathbf{x} = (x_1, \ldots, x_N) \to V_{\Lambda,\Lambda_0}(\mathbf{x})$ is well defined and lower bounded, so that also

$$H_{\Lambda,\Lambda_0}(\mathbf{x}) \equiv H_\Lambda(\mathbf{x}) + V_{\Lambda,\Lambda_0}(\mathbf{x}) \tag{2.7}$$

is lower bounded and measurable in $\mathbf{x}$. Thus the corresponding canonical partition function

$$Q_{\Lambda,\Lambda_0} \equiv (N!)^{-1} \int_{\Lambda^N} \exp[-\beta H_{\Lambda,\Lambda_0}(\mathbf{x})]\, dx \tag{2.8}$$

is well defined for all $\beta \geqslant 0$, is strictly positive, so that the corresponding free energy

$$f_{\Lambda,\Lambda_0} \equiv -(\beta|\Lambda|)^{-1}\ln Q_{\Lambda,\Lambda_0} \tag{2.9}$$

is well defined.

Let

$$d\mu_\Lambda \equiv (N!Q_\Lambda)^{-1}\exp[-\beta H_\Lambda(\mathbf{x})]\,d\mathbf{x} \tag{2.10a}$$

and

$$d\mu_{\Lambda,\Lambda_0} \equiv (N!Q_{\Lambda,\Lambda_0})^{-1}\exp[-\beta H_{\Lambda,\Lambda_0}(\mathbf{x})]\,d\mathbf{x} \tag{2.10b}$$

and denote by $\langle\ \rangle_0$ and $\langle\ \rangle_D$ expectations with respect to μ_Λ and μ_{Λ,Λ_0}, respectively. By Jensen's inequality we have, by introducing the definition of H_{Λ,Λ_0} into the one of Q_{Λ,Λ_0},

$$Q_{\Lambda,\Lambda_0} \geqslant Q_\Lambda \exp(-\beta\langle V_{\Lambda,\Lambda_0}\rangle_0) \tag{2.11}$$

On the other hand, again by Jensen's inequality applied to $Q_\Lambda/Q_{\Lambda,\Lambda_0}$,

$$Q_\Lambda = Q_{\Lambda,\Lambda_0} Q_\Lambda/Q_{\Lambda,\Lambda_0} \geqslant Q_{\Lambda,\Lambda_0}\exp(-\beta\langle V_{\Lambda,\Lambda_0}\rangle_D) \tag{2.12}$$

From (2.11) and the definitions (2.5) and (2.9) of f_Λ and f_{Λ,Λ_0}, respectively we get

$$f_{\Lambda,\Lambda_0} - f_\Lambda \leqslant |\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_0 \tag{2.13}$$

On the other hand, using (2.12) instead of (2.11), we get

$$f_{\Lambda,\Lambda_0} - f_\Lambda \geqslant |\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D \tag{2.14}$$

In order to control the limit of the right-hand sides of (2.13) and (2.14) as $\Lambda, \Lambda_0 \uparrow \mathbb{R}^2$, we introduce the corresponding one-particle correlation functions $g_\Lambda(x)$ and $g_{\Lambda,\Lambda_0}(x)$, $x \in \Lambda$:

$$g_\Lambda(x) \equiv [(N-1)!Q_\Lambda]^{-1}\int_{\Lambda^{N-1}}\exp[-\beta H_\Lambda(x_1,\ldots,x_{N-1},x)]\prod_{i=1}^{N-1}dx_i \tag{2.15}$$

and

$$g_{\Lambda,\Lambda_0}(x) \equiv [(N-1)!Q_{\Lambda,\Lambda_0}]^{-1} \times \int_{\Lambda^{N-1}}\exp[-\beta H_{\Lambda,\Lambda_0}(x_1,\ldots,x_{N-1},x)]\prod_{i=1}^{N-1}dx_i \tag{2.16}$$

Due to the lower bounds on H_Λ and H_{Λ,Λ_0} these functions are well-defined bounded measurable functions of x. Moreover, they are non-negative and have integral equal to $N = \rho|\Lambda|$, i.e.,

$$\int_\Lambda g_\Lambda(x)\,dx = \int_\Lambda g_{\Lambda,\Lambda_0}(x)\,dx = N = \rho|\Lambda| \tag{2.17}$$

In particular thus $g_\Lambda, g_{\Lambda,\Lambda_0} \in L^1(\Lambda, dx) \cap L^\infty(\Lambda, dx)$. From the definition of V_{Λ,Λ_0} and g_Λ, (2.6) and (2.15), respectively, we have

$$\langle V_{\Lambda,\Lambda_0}\rangle_0 = \int_\Lambda [\, g_\Lambda(x) - \rho\,] F_{\Lambda,\Lambda_0}(x)\,dx \tag{2.18}$$

where

$$F_{\Lambda,\Lambda_0}(x) \equiv \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \left[\varphi(x, x_l) - \rho \int_{C_l'} \varphi(x, y)\,dy \right], \qquad C_l' \equiv C_l \cap (\Lambda_0 - \Lambda) \tag{2.19}$$

C_l denotes the Wigner–Seitz cell with center x_l. In this derivation we have used that, for $x \in \Lambda$,

$$\int_{\Lambda_0 - \Lambda} \varphi(x, y)\,dy = \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \int_{C_l'} \varphi(x, z)\,dz \tag{2.20}$$

All integrals are well defined, due to the fact that $\varphi(x, y)$ is smooth for $y \neq x$. We remark also that

$$\int_{C_l'} \varphi(x, z)\,dz = \int_{C_0} \varphi(x, x_l + u)\,du \tag{2.21}$$

with

$$C_0 \equiv \left\{ u \in \mathbb{R}^2 \,|\, u = z - x_l \,,\, z \in C_l \cap (\Lambda_0 - \Lambda) \right\}$$

so that

$$F_{\Lambda,\Lambda_0}(x) = \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \psi(x, x_l) \tag{2.22}$$

with

$$\psi(x, x_l) \equiv \varphi(x, x_l) - \rho \int_{C_0} \varphi(x, x_l + u)\,du \tag{2.23}$$

In a similar way we obtain

$$\langle V_{\Lambda,\Lambda_0}\rangle_D = \int_\Lambda [\, g_{\Lambda,\Lambda_0}(x) - \rho\,] F_{\Lambda,\Lambda_0}(x)\,dx \tag{2.24}$$

Let now Λ and Λ_0 be squares in $\mathbb{R}^2$, centered at the origin, with sides parallel to the sides of the coordinate axis, with unit vectors e_i. The sides have lengths $(\sqrt{N}\,a)$ and $(\sqrt{N_0}\,a)$ for some fixed $a > 0$, and some integers N, N_0, with $N_0 > N$.

We take for simplicity N, N_0 such that $N = k^2$, $N_0 = k_0^2$, with k, k_0 integers. We choose the lattice $(\Lambda_0 - \Lambda)_d$ to be $[a\mathbb{Z}^2 \cap (\Lambda_0 - \Lambda)]$. In this case $C_0 = \{v \in \mathbb{R}^2 | v_i \leqslant a/2,\ i = 1,2\}$ and the distance from Λ to $(\Lambda_0 - \Lambda)_d$ is a. Let us denote by $\tilde{\Lambda}$ the set $\tilde{\Lambda} \equiv \{v \in \mathbb{R}^2 | v = v_1 + v_2,\ v_1 \in \Lambda,\ v_2 \in C_0\}$. We then have, using the translation invariance of φ,

$$\int_\Lambda dx \left[\int_{C_0} \varphi(x, x_l + u)\, du \right] = \int_\Lambda dx \left[\int_{C_0} \varphi(x - u, x_l)\, du \right]$$
$$= \int_{\tilde{\Lambda}} dy \left[\int_{C_0} \varphi(y, x_l)\, du \right] = a^2 \int_{\tilde{\Lambda}} dy\, \varphi(y, x_l)$$

Thus:

$$|\Lambda|^{-1} \int_\Lambda F_{\Lambda,\Lambda_0}(x)\, dx = |\Lambda|^{-1} \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \int_\Lambda \varphi(x, x_l)\, dx$$
$$- a^2 |\Lambda|^{-1} \rho \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \int_{\tilde{\Lambda}} \varphi(y, x_l)\, dy$$
$$= |\Lambda|^{-1} \sum_{x_l \in (\Lambda_0 - \Lambda)_d} \int_{\tilde{\Lambda} - \Lambda} \varphi(y, x_l)\, dy$$

if we choose $\rho = 1/a^2$. But for $|y - x_l| \leqslant 1$, $y \in \tilde{\Lambda} - \Lambda$ we have $|y - x_l| \geqslant a/2$, hence

$$\varphi(y, x_l) = -(2\pi)^{-1} \ln|y - x_l| \leqslant (2\pi)^{-1} \ln(a/2)$$

For $|y - x_l| \geqslant 1$, $y \in \tilde{\Lambda} - \Lambda$ we have

$$|y - x_l| \leqslant (\sqrt{2})^{-1} \left(a^2 \left(\sqrt{N_0} + \sqrt{N} \right)^2 - 2a \left(\sqrt{N} - \sqrt{N_0} \right) + 1 \right)^{1/2} \equiv A$$

thus in all cases

$$|\varphi(y, x_l)| \leqslant (2\pi)^{-1} \max(\ln a/2, \tfrac{1}{2} \ln A) \equiv c(N, N_0, a)$$

Hence

$$|\Lambda|^{-1} \left| \int_\Lambda F_{\Lambda,\Lambda_0}(x)\, dx \right| \leqslant |\Lambda|^{-1} \int_\Lambda |F_{\Lambda,\Lambda_0}(x)|\, dx$$
$$\leqslant (N_0 - N) c(N, N_0, a) \left(2a\sqrt{N} + 1 \right) / aN$$

From this we see that if $N, N_0 \to \infty$ in such a way that $(N_0 - N) c(N, N_0, a) (\sqrt{N})^{-1} \to 0$, then we have $\lim_{N \to \infty} |\Lambda|^{-1} \int_\Lambda |F_{\Lambda,\Lambda_0}(x)|\, dx = 0$. Hence we have proven the following lemma.

Lemma 2.1. Let $(\Lambda_0 - \Lambda)_d = [a\mathbb{Z}^2 \cap (\Lambda_0 - \Lambda)]$, with $a = 1/\rho$. Then if $N, N_0 \to \infty$ in such a way that $(N_0 - N) \ln(\sqrt{N_0} + \sqrt{N})/\sqrt{N} \to 0$, then

$$|\Lambda|^{-1} \int_\Lambda |F_{\Lambda,\Lambda_0}(x)|\, dx \to 0 \quad \blacksquare$$

Remark. The assumption about the convergence of N, N_0 to ∞ is satisfied, e.g., when $N_0 = N + C_2 N^\alpha$, $\alpha < 1/2$, as $N \to \infty$.

Using this lemma it follows from (2.18) and (2.24) that $\lim |\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_X$ exists, where X stands for 0 or D, iff the corresponding limit of $|\Lambda|^{-1}\int_\Lambda g_Y(x) F_{\Lambda,\Lambda_0}(x)\,dx$ exists, where $g_Y = g_\Lambda$ or g_{Λ,Λ_0}.

Suppose now that g_Λ is in $L^\infty(dx)$, i.e., it is essentially bounded. Then we have

$$|\Lambda|^{-1}\left|\int_\Lambda g_\Lambda(x) F_{\Lambda,\Lambda_0}(x)\,dx\right| \leqslant \|g_\Lambda\|_\infty |\Lambda|^{-1}\int_\Lambda |F_{\Lambda,\Lambda_0}(x)|\,dx$$

where $\|g_\Lambda\|_\infty$ is the L^∞-norm. The right-hand side converges to zero under the assumption that $\|g_\Lambda\|_\infty (N_0 - N)\ln(\sqrt{N_0} + \sqrt{N})/\sqrt{N} \to 0$, as one sees from the proof of Lemma 2.1.

In the same way one sees that the same holds with g_Λ replaced by g_{Λ,Λ_0}. Thus we get from (2.18), (2.24) that $|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_0 \to 0$, and the same with $\langle\ \rangle_0$ replaced by $\langle\ \rangle_D$. From (2.13), (2.14) we get then the following theorem.

Theorem 2.2. Let Λ, Λ_0 be concentric squares of side lengths $(N/\rho)^{1/2}$ and $(N_0/\rho)^{1/2}$, respectively, where $\rho > 0$ is the given density. Let Dobrushin boundary conditions be given on $(\rho)^{-1/2}\mathbb{Z}^2 \cap (\Lambda_0 - \Lambda)$, i.e., on the points of the lattice $\rho^{-1/2}\mathbb{Z}^2$ that belong to the complement of Λ in Λ_0. Suppose that the one-particle correlation functions $g_\Lambda(x)$ and $g_{\Lambda,\Lambda_0}(x)$ are bounded for a.e. x in such a way that as $N_0 \geqslant N \to \infty$ one has $\|h\|_\infty = o_h(\sqrt{N}/(N_0 - N)\ln(\sqrt{N_0} + \sqrt{N})])$, where h stands for g_Λ or g_{Λ,Λ_0}, and we use the notation $o_h(x)$ to say that $o_h(x)$ is a function, depending on h, such that $o_h(x)/|x| \to 0$ as $|x| \to \infty$. Then the free energy f_Λ for free boundary conditions and the free energy f_{Λ,Λ_0} for Dobrushin boundary conditions on $\rho^{-1/2}\mathbb{Z}^2 \cap (\Lambda_0 - \Lambda)$ converge to the same limit as $\Lambda_0 \supset \Lambda \uparrow \mathbb{R}^2$.

Remark. The assumptions on g_Λ, g_{Λ,Λ_0} are, e.g., satisfied if these functions are uniformly in L^∞ as $N_0 \geqslant N \to \infty$ in such a way that $(N_0 - N)\ln(\sqrt{N_0} + \sqrt{N})/\sqrt{N} \to 0$.

3. THE COULOMB PLASMA IN CIRCULAR DOMAINS

Let Λ and Λ_0 be the inner of the circles $\Lambda \equiv \{x \in \mathbb{R}^2 \,|\, |x| < R\}$ and $\Lambda_0 \equiv \{x \in \mathbb{R}^2 \,|\, |x| < R_0\}$, respectively, for some $R_0 > R > 0$. Let Y be any discrete subset of $\Lambda_0 - \Lambda$ (below we shall take Y to be a lattice). Let f_Λ be defined by (2.5). Define, correspondingly as in (2.6),

$$V_{\Lambda,\Lambda_0}(\mathbf{x}) \equiv \sum_{i=1}^{N} \sum_{x\in Y} \varphi(x,x_i) - \rho \sum_{x\in Y} \int_{\Lambda} \varphi(y,x)\,dy - \rho \sum_{i=1}^{N} \int_{\Lambda_0-\Lambda} \varphi(y,x_i)\,dy$$

$$+ \rho^2 \int_{\Lambda} \left[\int_{\Lambda_0-\Lambda} \varphi(x,y)\,dy \right] dx, \qquad \mathbf{x} = (x_1, \ldots, x_N) \tag{3.1}$$

Set, moreover, $H_{\Lambda,\Lambda_0} \equiv H_\Lambda + V_{\Lambda,\Lambda_0}$, with H_Λ given by (2.1), and define f_{Λ,Λ_0} by (2.8), (2.9).

In the following we shall need the general mean-value theorem:

Lemma 3.1. For any ball Λ on $\mathbb{R}^2$, any $h \in L^1(\Lambda, dx)$ with $h(x) = h(|x|)$ and any $y \notin \Lambda$ one has

$$\int_\Lambda h(|x|)\varphi(x,y)\,dx = \left[\int_\Lambda h(|x|)\,dx \right] \varphi(y)$$

where $\varphi(x,y) = -(2\pi)^{-1}\ln|x-y|$ is the logarithmic potential, and we use the notation $\varphi(0,y) \equiv \varphi(y)$.

Proof. This is a well-known result following from the fact that $\varphi(x,y)$ is harmonic for $x \neq y$. ■

Using this lemma we can easily prove the following lemma.

Lemma 3.2. For circular domains Λ, Λ_0 one has $\langle V_{\Lambda,\Lambda_0}\rangle_0 = 0$ and $f_{\Lambda,\Lambda_0} \leqslant f_\Lambda$.

Proof. As in Section 2, we prove easily

$$f_{\Lambda,\Lambda_0} - f_\Lambda \leqslant |\Lambda|^{-1} \langle V_{\Lambda,\Lambda_0}\rangle_0 \tag{3.2}$$

Hence it certainly suffices to prove $\langle V_{\Lambda,\Lambda_0}\rangle_0 = 0$. From the definition (2.15) of g_Λ we have that $g_\Lambda(x)$ is invariant under rotations, since $H_\Lambda(\mathbf{x})$ is invariant under the rotation of all the x_i in $\mathbb{R}^2$ and the integration domain Λ^N in the definition of g is also rotation invariant, Λ being here a circle.

Using the fact that

$$\int_\Lambda g_\Lambda\,dx = \rho|\Lambda| \tag{3.3}$$

together with Lemma 3.1, the lemma follows using the analog of the formula (2.18), namely,

$$\langle V_{\Lambda,\Lambda_0}\rangle_0 = \int_\Lambda [\, g_\Lambda(x) - \rho\,] F_{\Lambda,\Lambda_0}(x)\,dx \tag{3.4}$$

with

$$F_{\Lambda,\Lambda_0}(x) \equiv \sum_{y\in Y} \varphi(x,y) - \rho \int_{\Lambda_0-\Lambda} \varphi(x,y)\,dy \tag{3.5}$$

In fact

$$\int_{\Lambda} g_{\Lambda}(x) F_{\Lambda,\Lambda_0}(x)\, dx = \rho|\Lambda| \sum_{y\in Y} \varphi(y) - \rho^2|\Lambda| \int_{\Lambda_0-\Lambda} \varphi(y)\, dy \tag{3.6}$$

where we used (3.5), (3.3), and Lemma 3.1, and similarly

$$-\int_{\Lambda} F_{\Lambda,\Lambda_0}(x)\, dx = -|\Lambda| \sum_{y\in Y} \varphi(y) + \rho|\Lambda| \int_{\Lambda_0-\Lambda} \varphi(y)\, dy$$

Thus $\langle V_{\Lambda,\Lambda_0}\rangle_0 = 0$ and the lemma is proven. ■

Lemma 3.3. Let Λ, Λ_0 be concentric circular domains as in Lemma 3.2 and let, as in Section 2, $\langle V_{\Lambda,\Lambda_0}\rangle_D$ be given by (2.6). Then

$$\langle V_{\Lambda,\Lambda_0}\rangle_D = \sum_{y\in Y} \int_{\Lambda} (g_{\Lambda,\Lambda_0}(x) - \rho)\varphi(x, y)\, dx$$

and

$$|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D = |\Lambda|^{-1} \int_{\Lambda} g_{\Lambda,\Lambda_0}(x) \sum_{y\in Y} [\varphi(x-y) - \varphi(y)]\, dx$$

Proof. We use the representation (2.18) of $\langle V_{\Lambda,\Lambda_0}\rangle_D$, i.e.,

$$\langle V_{\Lambda,\Lambda_0}\rangle_D = -\int_{\Lambda} dx\, [g_{\Lambda,\Lambda_0}(x) - \rho] F_{\Lambda,\Lambda_0}(x) \tag{3.7}$$

with F_{Λ,Λ_0} given by (3.6). We now prove that (as in the rectangular case) the contribution to (3.7) from the background is vanishing, because of symmetry reasons. In fact

$$\int_{\Lambda} dx\, [g_{\Lambda,\Lambda_0}(x) - \rho] \int_{\Lambda_0-\Lambda} \varphi(x, y)\, dy = \int_{\Lambda} dx\, [g_{\Lambda,\Lambda_0}(x) - \rho] \int_{\Lambda_0-\Lambda} \varphi(y)\, dy \tag{3.8}$$

where we used the fact that

$$\int_{\Lambda_0-\Lambda} \varphi(x, y)\, dy = \int_{\Lambda_0-\Lambda} \varphi(y)\, dy$$

by the invariance properties of φ and the domains. But $\int_{\Lambda} dx\, [g_{\Lambda,\Lambda_0}(x) - \rho] = 0$, hence (3.8) vanishes, $\int_{\Lambda_0-\Lambda} \varphi(y)\, dy$ being independent of x. This proves the first equality in the lemma.

To prove the second one we only need to remark that, using the first one, we have

$$\begin{aligned}\langle V_{\Lambda,\Lambda_0}\rangle_D &= \sum_{y\in Y} \int_{\Lambda} [g_{\Lambda,\Lambda_0}(x) - \rho]\varphi(x, y)\, dx \\ &= \sum_{y\in Y} \int_{\Lambda} g_{\Lambda,\Lambda_0}(x)\varphi(x-y)\, dx - \rho \sum_{y\in Y} \int_{\Lambda} \varphi(x, y)\, dx \end{aligned} \tag{3.9}$$

But for the second term we have, using Lemma 3.1,

$$\rho \sum_{y\in Y}\int_{\Lambda}\varphi(x,y)\,dx=\rho\sum_{y\in Y}|\Lambda|\varphi(y)=\Big[\int_{\Lambda}g_{\Lambda,\Lambda_0}(x)\,dx\Big]\sum_{y\in Y}\varphi(y) \quad (3.10)$$

where in the latter inequality we have used (2.17). Introducing (3.10) in (3.9) we get the lemma. ■

Remark. We believe it should be possible to show from Lemma 3.3 that $\lim_{\Lambda,\Lambda_0\uparrow\mathbb{R}^2}|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D=0$, for a given lattice structure Y in $\Lambda_0-\Lambda$, e.g., triangular or cubic (see also the remark at the end of this section). In the following we shall, however, restrict our considerations to special discrete sets Y which have nice discrete rotation-invariant properties.

Let us consider circles C_i of radii R_i, $i=1,2,\ldots$ centered at the origin. Take on C_i equidistantly placed N_i charges $x_l^{(i)}$, $l=1,\ldots,N_i$. Let as before Λ be the inner of the circle with center at the origin and radius $R<R_i$. Let Λ_0 be the concentric circle with radius $R_0=R_{i_0}$ for some i_0. We have the following lemma.

Lemma 3.4. Let φ be the logarithmic potential and let $x_l^{(i)}$ be equidistant points on the circumference C_i, $l=1,\ldots,N_i$. Then for any $N_i\in\mathbb{N}$

$$\sum_{l=1}^{N+i}\varphi\big(|x-x_l^{(i)}|/|x_l^{(i)}|\big)=\tfrac{1}{2}\varphi\big(1+(|x|/R_i)^{N+i}\big) \quad (3.11)$$

with the notation $\varphi(|x|)\equiv\varphi(x)$.

Proof. We have with $y\equiv|x|/R_i=|x|/|x_l^{(i)}|$

$$\sum_l\varphi\big(|x-x_l^{(i)}|/|x_l^{(i)}|\big)=\sum_l\frac{1}{2}\varphi\big[1+y^2-2y\cos(\varphi_l-\varphi_0)\big]$$

$$=\frac{1}{2}\varphi\Big(\prod_l\big[y-e^{i(\varphi_l-\varphi_0)}\big]\big[y-e^{-i(\varphi_l-\varphi_0)}\big]\Big)$$

where we have used the angles φ_l, φ_0 of the vectors $x_l^{(i)}$ and x, respectively, with respect to some given reference axis. But

$$\prod_{l=1}^{N+i}\big[y-e^{i(\varphi_l-\varphi_0)}\big]\big[y-e^{-i(\varphi_l-\varphi_0)}\big]=y^{N+1}+1 \quad (3.12)$$

From (3.11), (3.12) we then get the lemma. ■

Remark. It might be amusing to note that the right-hand side of the inequality in Lemma 3.4 is minus the free energy of the one-dimensional Ising model with periodic boundary conditions, for coefficients β,J such that th $\beta J=|x|/R_i$.

We shall now use Lemma 3.4 to estimate the quantity $|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D$ in Lemma 3.3. We get, splitting the sum $\sum_{y\in Y}$ with $Y\equiv\{x_l^{(i)},\ l=$

$1, \ldots, N+i; i=1,2, \ldots$

$$|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D = |\Lambda|^{-1}\sum_{i=1}^{i_0}\sum_{l=1}^{N_i}\int_\Lambda g_{\Lambda,\Lambda_0}(x)\left[\varphi(x-x_l^{(i)})-\varphi(x_l^{(i)})\right]dx$$
$$= |\Lambda|^{-1}\sum_{i=1}^{i_0}\sum_{l=1}^{N_i}\int_\Lambda g_{\Lambda,\Lambda_0}(x)\varphi(|x-x_l^{(i)}|/|x_l^{(i)}|)\,dx \quad (3.13)$$

where we used the property $\varphi(x)-\varphi(y)=\varphi(|x|,|y|)$ of the logarithmic potential. Using Lemma 3.4, (3.13) is equal to

$$(2|\Lambda|)^{-1}\sum_{i=1}^{i_0}\int_\Lambda g_{\Lambda,\Lambda_0}(x)\phi\left[1+(|x|/R_i)^{N_i}\right]dx$$

Hence we have proven the following:

Lemma 3.5. Let Λ, Λ_0 be concentric circular domains, $\Lambda_0 \supset \Lambda$ with center at the origin. Assume in $\Lambda_0 - \Lambda$ is given a regular configuration of positive charges as described above. Let $\langle V_{\Lambda,\Lambda_0}\rangle_D$ be given by (2.6), with $(\Lambda_0-\Lambda)_d$ the points where the positive charges are situated. Then

$$|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D = (2|\Lambda|)^{-1}\sum_{i=1}^{i_0}\int_\Lambda g_{\Lambda,\Lambda_0}(x)\varphi\left[1+(|x|/R_i)^{N_i}\right]dx \quad \blacksquare \quad (3.14)$$

From this lemma we can now control easily the limit $\Lambda_0 \supset \Lambda \uparrow \mathbb{R}^2$. We have, namely, for all $x \in \Lambda$

$$\varphi\left[1+(|x|/R_i)^{N_i}\right] = -(2\pi)^{-1}\ln\left[1+(|x|/R_i)^{N_i}\right]$$
$$\geqslant -(2\pi)^{-1}(|x|/R_i)^{N_i} \quad (3.15)$$

where we used the definition of φ and the inequality $-\ln(1+q) \geqslant -q$ for all $q \geqslant 0$. Assume now that for

$$\rho|\Lambda| = N \in \mathbb{N} \quad (3.16)$$

i.e.,

$$R = (N/\rho\pi)^{1/2} \quad (3.17)$$

$$g_{\Lambda,\Lambda_0}(x) \leqslant C(N) \quad (3.18)$$

for a.a. x and some constant $C(N)$.

Then introducing this and (3.15) into the integral in (3.14) we obtain with $R_i = R + a_i$, $a_i > 0$, $i = 1,2,\ldots$

$$\int_\Lambda g_{\Lambda,\Lambda_0}(x)\varphi\left[1+(|x|/R_i)^{N_i}\right]dx \geqslant -\frac{1}{2\pi}C(N)\int_\Lambda(|x|/R_i)^{N_i}dx$$
$$= -C(N)R^2(R/R_i)^{N_i}(N_i+2)^{-1}$$

and thus

$$|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D \geqslant -\frac{1}{2|\Lambda|} C(N)R^2 \sum_{i=1}^{i_0} (N_i+2)^{-1}\left[R/(R+a_i)\right]^{N_i}$$

Taking $i_0 \to \infty$ yields

$$\lim_{\Lambda_0 \uparrow \mathbb{R}^2} |\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D \geqslant -(2|\Lambda|)^{-1} C(N)R^2 \sum_{i=1}^{\infty} (N_i+2)^{-1}\left[R/(R+a_i)\right]^{N_i}$$

and with (3.16) and (3.17)

$$\lim_{\Lambda \uparrow \mathbb{R}^2} \lim_{\Lambda_0 \uparrow \mathbb{R}^2} |\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D \geqslant -\lim_{N\to\infty} \frac{C(N)}{2\pi} \sum_{i=1}^{\infty} (N_i+2)^{-1} \times\left[\left(\frac{N}{\rho\pi}\right)^{1/2} \Big/ \left(\left(\frac{N}{\rho\pi}\right)^{1/2} + a_i\right)\right]^{N_i} \tag{3.19}$$

As before we now assume that $C(N) = O(\sqrt{N})$. Then if we first choose $a_i = i$ and impose neutrality outside of Λ, then as one easily computes, setting for simplicity $\rho\pi = 1$, we have that

$$N_i = 2\sqrt{N} + 2i - 1$$

Then we can estimate in (3.19) using $q \leqslant e^{q-1}$ for $0 \leqslant q \leqslant 1$

$$\sum_{i=1}^{\infty} (N_i+2)^{-1}\left[\sqrt{N}/(\sqrt{N}+i)\right]^{2(\sqrt{N}+i)-1} \leqslant N^{-1/2} \sum_{i=1}^{\infty} e^{-2i+i/(\sqrt{N}+i)}$$
$$\leqslant N^{-1/2} e(1/1-e^{-2})$$

so that

$$\lim_{\Lambda \uparrow \mathbb{R}^2} \lim_{\Lambda_0 \uparrow \mathbb{R}^2} |\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D = 0 \tag{3.20}$$

Let us consider the case where we do not impose charge neutrality outside of Λ and take $a_i = i$ and $N_i = \sqrt{N}$. Then we estimate in (3.19)

$$\sum_{i=1}^{\infty} (\sqrt{N}+2)^{-1}\left[\sqrt{N}/(\sqrt{N}+i)\right]^{\sqrt{N}} \leqslant (\sqrt{N}+2)^{-1} \int_0^{\infty} \left[\sqrt{N}/(\sqrt{N}+x)\right]^{\sqrt{N}} dx$$
$$= (\sqrt{N}+2)^{-1}\left[\sqrt{N}/(\sqrt{N}-1)\right]$$

where we used that $(\sqrt{N}/(\sqrt{N}+i))^{\sqrt{N}}$ is strictly decreasing with i. Thus (3.20) follows with $C(N) = O(\sqrt{N})$.

We summarize the above discussion in the following theorem.

Theorem 3.6. Let N be an integer and Λ be the circle with center at the origin of radius $R = (N/\rho\pi)^{1/2}$. Let a_i be any sequence of positive numbers and $R_i = R + a_i$, $i = 1,2, \ldots,$ and let C_i be the circle of center the origin and radius R_i. Take on C_i equidistantly placed N_i charges. Let Λ_0 be the concentric circle of radius R_i for some i_0. Let f_Λ and f_{Λ,Λ_0} be defined as in Section 2. Then, under the assumption $g_{\Lambda,\Lambda_0}(x) \leqslant C(N)$ for some constant $C(N)$, we get

$$\lim_{\Lambda\uparrow\mathbb{R}^2} \lim_{\Lambda_0\uparrow\mathbb{R}^2} f_{\Lambda,\Lambda_0} = \lim_{\Lambda\uparrow\mathbb{R}^2} f_\Lambda$$

if N_i, a_i, and $C(N)$ are such that

$$\lim_{N\to\infty} C(N) \sum_{i=1}^{\infty} (N_i + 2)^{-1} \left\{ (N/\rho\pi)^{1/2} / \left[(N/\rho\pi)^{1/2} + a_i \right] \right\}^{N_i} = 0$$

The latter assumption is, e.g., verified when $C(N) = O(\sqrt{N})$, $\rho\pi = 1$, $a_i = i$ and in addition $N_i = 2\sqrt{N} + 2i - 1$ or $N_i = \sqrt{N}$.

Remark. The configuration in $\Lambda_0 - \Lambda$ ("boundary condition") for which Proposition 3.1 holds corresponds to a "disturbed" crystal. A corresponding study of a triangular or square lattice configuration involves nice number theoretical problems concerning the distribution of lattice points in circles; see, e.g., Ref. 22.

Remark. The assumptions in Theorem 3.6 concerning g_{Λ,Λ_0} are of the same type as those in Theorem 2.2. Therefore the remarks following Theorem 2.2 hold also here.

4. THE COULOMB PLASMA IN CYLINDRICAL DOMAINS

Let now $\Lambda \equiv \Lambda_N$ be the surface of the cylinder $S^1 \times [-N - 1/2, N + 1/2]$, for some integer N, S^1 being a circle of radius $R = M/2\pi$, for some integer M. Let $\Lambda_0 \equiv \Lambda_{N_0}$ be $S^1 \times [-N_0 - 1/2, N_0 + 1/2]$, for some integer $N_0 > N$. Let us choose Cartesian coordinates (x_1, x_2) with the origin at the center of symmetry of Λ, such that we have a periodicity of period M in the x_2 direction. The corresponding Coulomb potential $\varphi(x, y)$, with free boundary conditions, is given by

$$\begin{aligned}\varphi(x, y) &\equiv M^{-1} \int_{\mathbb{R}^1} dq_1 \sum_{q_2 = 2\pi n/M} 2\pi \left(q_1^2 + q_2^2\right)^{-1} \\ &= -\frac{\pi}{M} |x_1 - x_2| - \frac{1}{2} \ln\left[1 + \exp\left(-\frac{4\pi}{M} |x_1 - y_1| \right) \right. \\ &\qquad \left. - 2 \exp\left(-\frac{2\pi}{M} |x_1 - y_1| \right) \cos \frac{2\pi}{M} (x_2 - y_2) \right]\end{aligned} \tag{4.1}$$

where,

$$0 \leqslant x_2, \qquad y_2 \leqslant M, \qquad -N - 1/2 \leqslant x_1, \qquad y_1 \leqslant N + 1/2$$

Let us consider a square lattice configuration Y in $\Lambda_0 - \Lambda$ of positive charges, i.e., we place positive unit charges at the points $(x_{1,i}, x_{2,j})$ with

$$x_{1,i} = \pm(N+i)_1 \qquad i = 1, \ldots, N_0 - N, \qquad x_{2,j} = j, \qquad j = 1, \ldots, M$$

We can then derive similarly as in Section 3, that, for symmetry reasons,

$$\lim_{\Lambda_0 \uparrow \mathbb{R}^2} V_{\Lambda,\Lambda_0} = 0 \tag{4.2}$$

from which it then follows, as in Section 3,

$$\lim_{\Lambda_0 \uparrow \mathbb{R}^2} f_{\Lambda,\Lambda_0} - f_\Lambda \leqslant 0 \tag{4.3}$$

Moreover we get similarly as in Section 3

$$\langle V_{\Lambda,\Lambda_0} \rangle_D = -\frac{1}{2} \int g_{\Lambda,\Lambda_0}(x) \sum_{y \in Y} \ln\left[1 + \xi^2 - 2\xi \cos\frac{2\pi}{M}(x_2 - y_2)\right] \tag{4.4}$$

with $\xi \equiv \exp[-(2\pi/M)|x_1 - y_1|]$, where we used the reflection symmetry of $g_{\Lambda,\Lambda_0}(x)$ in the x_1 coordinate to cancel the contribution of the term $-(\pi/M)|x_1 - y_1|$ in the potential. Now we compute the sum over Y as $\sum_{i=N-N_0}^{N_0-N} \sum_{j=1}^{M}$. We remark that, calling ξ_i the quantity defined as ξ with $y_1 = N + i$, we have

$$\sum_{j=1}^{M} -\frac{1}{2} \ln\left[1 + \xi_i^2 - 2\xi_i \cos\frac{2\pi}{M}(x_2 - y_{2,j})\right]$$
$$= -\frac{1}{2}\ln\left(1 + \xi_i^{2M} + 2\xi_i^M \cos 2\pi x_2\right) \geqslant -\xi_i^M$$

Introducing this into (4.4) we get

$$\langle V_{\Lambda,\Lambda_0} \rangle_D = -\int_\Lambda dx\, g_{\Lambda,\Lambda_0}(x) \sum_{i=1}^{N_0-N} \left(e^{-2\pi|x_1-(N+i)|} + e^{-2\pi|x_1+(N+i)|}\right)$$
$$\geqslant -\int_\Lambda g_{\Lambda,\Lambda_0}(x) e^{-2\pi N}\Big[e^{2\pi x_1}\left(1 - e^{-2\pi(N_0-N)}\right)\left(1 - e^{-2\pi}\right)^{-1}$$
$$+ e^{-2\pi x_1}\left(1 - e^{-2\pi(N_0-N)}\right)\left(1 - e^{-2\pi}\right)^{-1}\Big] \tag{4.5}$$

Let us now assume the bound $\| g_{\Lambda,\Lambda_0}\|_\infty \leqslant C(N)$. Introducing this into (4.5) we get

$$|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D \geqslant -C'(N)e^{-2\pi N}/(1-e^{-2\pi})$$
$$\times\int_\Lambda \left[e^{2\pi x_1}(1-e^{-2\pi(N_0-N)})+e^{-2\pi x_1}(1-e^{-2\pi(N_0-N)})\right]dx,$$
$$C'(N)\equiv C(N)/M(2N+1)$$

hence

$$\lim_{N_0\to\infty}|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D \geqslant -e^{-2\pi N}(1-e^{-2\pi})^{-1}\frac{M}{2\pi}\left(e^{2\pi(N+1/2)}-e^{-2\pi(N+1/2)}\right)$$

and thus

$$\lim_{N\to\infty}\lim_{N_0\to\infty}|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D = 0$$

This then yields, similarly as in Section 3

$$\lim_{N\to\infty}\lim_{N_0\to\infty} f_{\Lambda,\Lambda_0}-f_\Lambda \geqslant 0$$

which together with (4.3), yields

$$\lim_{N\to\infty}\lim_{N_0\to\infty} f_{\Lambda,\Lambda_0}=\lim_{N\to\infty} f_\Lambda$$

Hence we have the following theorem.

Theorem 4.1. Let Λ,Λ_0 be the surfaces of cylinders $S_R^1\times[-N-1/2,N+1/2]$ respectively $S_R^1\times[-N_0-1/2,N_0+1/2]$, with integers $N_0>N$ and S_R^1 the circle with center at the origin and radius $R=M/2\pi$ for some integer M. Let f_Λ and f_{Λ,Λ_0} be, respectively, the canonical free energy with free boundary conditions, in Λ and the Dobrushin boundary conditions in $\Lambda_0-\Lambda$, consisting of placing unit positive charges forming the square crystalline configuration Y described above. Then the bound $\|g_{\Lambda,\Lambda_0}\|_\infty=o(N)$ implies

$$\lim_{N\to\infty}\lim_{N_0\to\infty} f_{\Lambda,\Lambda_0}=\lim_{N\to\infty} f_\Lambda$$

Remark. Instead of the above square crystalline configuration we could have taken as well a smooth positive charge distribution in the x_2-direction, for example $\rho(x_1,x_2)=2\rho\sin^2(\pi x_2)$. In a similar way as above one arrives at the formula

$$\langle V_{\Lambda,\Lambda_0}\rangle_D=\int_\Lambda (g_{\Lambda,\Lambda_0}(x)-\rho)F_{\Lambda,\Lambda_0}(x)\,dx$$

with

$$F_{\Lambda,\Lambda_0}(x) \equiv -\frac{1}{2}\int_0^M d\eta \sum_{y\in Y} \ln\left[1+\xi_y^2 - 2\xi_y\cos\frac{2\pi}{M}(\eta - y_2)\right] \times\left[2\rho\sin^2\pi\eta - \rho\right]$$

with

$$\xi_y \equiv \exp\left(-\frac{2\pi}{M}|x-y|\right)$$

Integration by parts gives

$$F_{\Lambda,\Lambda_0}(x) = -\frac{1}{2}\frac{\rho}{M}\sum_{y\in Y}\int_0^M d\eta$$

$$\times 2\xi_y \sin\frac{2\pi}{M}(\eta - y_2)\sin 4\pi\eta \Big/ \left[1+\xi_y^2 - 2\xi_y\cos\frac{2\pi}{M}(\eta - y_2)\right]$$

$$= \frac{1}{2}\rho\sum_{y\in Y}\xi_y^{N_0}\cos(N_0 M)$$

Hence

$$\lim_{N_0\to\infty} F_{\Lambda,\Lambda_0}(x) = \frac{\rho}{2}\sum_{n=-\infty}^{\infty}\cos(N_0 M)\exp\left[-\frac{2\pi}{M}\left(R+\frac{1}{2}+|n|\right)\right] \times 2\cosh\left(\frac{2\pi}{M}N\right)$$

$$= \frac{\rho}{2}\cos(2\pi x_2)(1-e^{-2\pi})^{-1}\exp\left[-2\pi\left(R+\frac{1}{2}\right)\right]\cosh(2\pi|x|)$$

Using the bound $\| g_{\Lambda,\Lambda_0}\|_\infty = o(N)$, see Ref. 18, we deduce that

$$\lim_{N\to\infty}\lim_{N_0\to\infty}|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_D = 0$$

Similarly as in Theorem 4.1 one proves

$$\lim_{N\to\infty}\lim_{N_0\to\infty}|\Lambda|^{-1}\langle V_{\Lambda,\Lambda_0}\rangle_0 = 0$$

and thus Theorem 4.1 extends to the case of the present Dobrushin condition.

Remark. The assumptions in Theorem 4.1 are of the same type as in Theorem 2.2 and Theorem 3.7. Thus the same remarks on when the assumptions on g_Λ, g_{Λ,Λ_0} are satisfied hold. See also Ref. 18.

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REFERENCES

1. Th. Choquard, Selected topics on the equilibrium mechanics of Coulomb systems, pp. 347–407, in Gabor Kalman, P. Carini, eds., *Strongly Coupled Plasmas* (Plenum Press, New York, 1978).
2. E. H. Lieb and H. Narnhofer, *J. Stat. Phys.* **12**:291–310 (1975).
3. R. R. Sari and D. Merlini, *J. Stat. Phys.* **14**:91–100 (1976).
4. E. P. Wigner, *Trans. Faraday Soc.* **34**:678 (1938).
5. R. R. Sari, D. Merlini, and R. Calinon, *J. Phys. A* **9**:1539–1551 (1976).
6. J. P. Hansen and E. Pollock, *Phys. Rev. A* **8**:3110 (1973).
7. H. Kunz, *(Leipzig) Ann. Phys.* **85**:303–335 (1974).
8. R. B. Griffiths, pp. 7–109 in *Phase Transitions and Critical Phenomena*, Vol. 1, Ed. C. Domb, M. S. Green (Academic Press, London-New York, 1972).
9. M. Fisher and J. Lebowitz, *Commun. Math. Phys.* **19**:251–272 (1970).
10. F. Guerra, L. Rosen, and B. Simon, *Ann. Inst. H. Poincaré* **15**:231–334 (1976).
11. S. Albeverio and R. Høegh-Krohn, *Commun. Math. Phys.* **68**:95–128 (1979).
12. J. Glimm, A. Jaffe, *Quantum Physics* (Springer, Berlin, 1981).
13. O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics* (Springer, Berlin, 1981).
14. R. Calinon, Ph. Choquard, D. Jamin, and J. Navet, in *Ordering in Two-Dimensions*, Ed. S. K. Sinha (North Holland, Amsterdam, 1980).
15. A. Alastuey and B. Jancovici, *J. Stat. Phys.* **24**:443–449 (1981).
16. Ch. Gruber and Ph. Martin, *Ann. Phys.* (*N.Y.*) **131**:56 (1981).
17. Ph. Choquard, private Communication
18. S. Albeverio, F. Martinelli, and D. Merlini, in preparation.
19. F. Martinelli and D. Merlini, in preparation.
20. S. Chakravarty and C. Dasgupta, *Phys. Rev. B* **22**:369 (1980).
21. B. Jancovici, *Phys. Rev. Lett.* **46**: 386–388 (1981).
22. F. Fricker, *Einführung in die Gitterpunktlehre* (Birkhäuser, Basel, 1982).